Econ 802

Answers to Midterm #1

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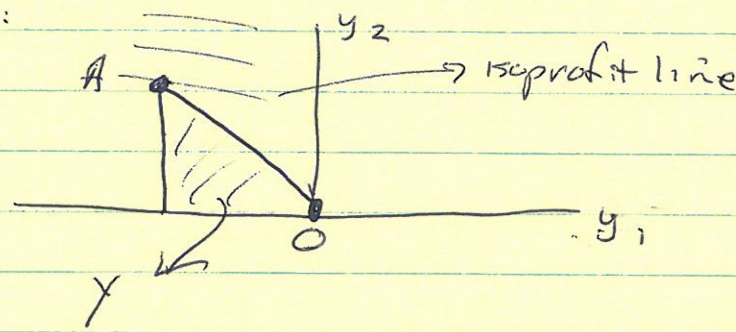
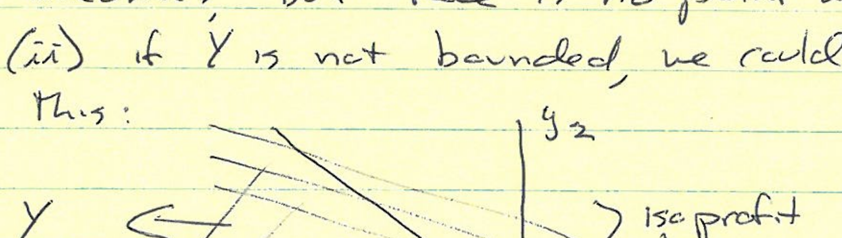
1 (a) If Y is closed and bounded then it is compact. The function py (profit) is continuous in y. The Weierstrasse Theorem says that a continuous function defined on a compact set reaches a maximum value at some point in the set. Therefore the profit max problem has a solution.

(i) if Y is not closed, we could have a situation like this:



where max profit would occur at point A but A is not in the set Y. The closer we get to A, the higher profit becomes, but there is no point where it is maximized.

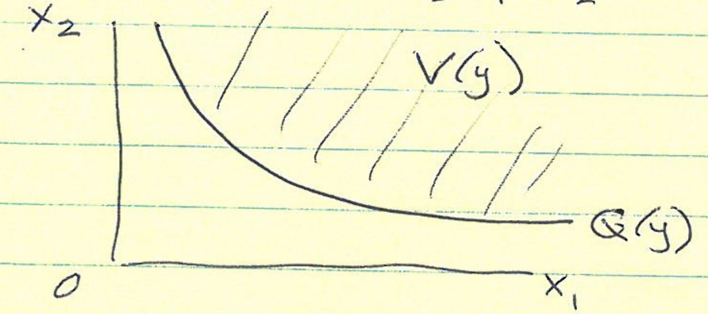
(ii) if Y is not bounded, we could have a situation like this:



where profit can always be increased by moving further out along the upper boundary of Y.

(b) To show that $V(y)$ is convex, consider the technical rate of substitution, which is the absolute value of the slope of the isoquant. We have

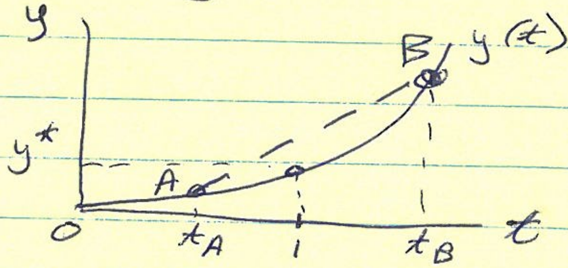
$$TRS = \frac{MP_1}{MP_2} = \frac{a x_1^{a-1} x_2^{b}}{b x_1^{a} x_2^{b-1}} = \frac{a}{b}\frac{x_2}{x_1}.$$ Thus as we move along an isoquant, the slope becomes flatter as $\frac{x_2}{x_1}$ falls, so $V(y)$ is convex (in fact, strictly)



To show that Y may not be convex choose some arbitrary point $x^* = (x_1^*, x_2^*) > 0$ with $y^* = (x_1^*)^a (x_2^*)^b$.
Write $y(t) = f(tx^*) = t^{a+b} f(x^*) = t^{a+b} y^*$.
We have $y'(t) = (a+b)t^{a+b-1} y^* > 0$

$$y''(t) = (a+b)(a+b-1)t^{a+b-2} y^*$$

If $a+b > 1$ (increasing returns) then we have a graph like this



points A and B are feasible but points on the dashed line segment between A and B ~~[illegible]~~ are not. So Y is not convex.

(c) Write $g(x) \equiv pf(x) - wx$.
Because f is strictly concave and wx is linear, g is strictly concave.
Suppose x^* and x^{**} both max $g(x)$ and $x^* \neq x^{**}$. Then by strict concavity there is some $x' = tx^* + (1-t)x^{**}$ with $0 < t < 1$ such that $g(x') > tg(x^*) + (1-t)g(x^{**})$. But this contradicts the fact that $g(x^*) = g(x^{**})$ and both x^* and x^{**} achieve the maximum profit. So there cannot be two distinct solutions, implying that any solution must be unique.

2 (a) $\max\ px^a - wx \Rightarrow$ FOC: $apx^{a-1} = w$

$$\Rightarrow x^{a-1} = \frac{w}{ap}$$

(note: SOC here is sufficient) $\Rightarrow x = \left(\frac{ap}{w}\right)^{\frac{1}{1-a}}$

$$\Rightarrow \pi(p,w) = p\left(\frac{ap}{w}\right)^{\frac{a}{1-a}} - w\left(\frac{ap}{w}\right)^{\frac{1}{1-a}}$$

$$\Rightarrow \pi(p,w) = p^{\frac{1}{1-a}} w^{-\frac{a}{1-a}} A$$ where $A \equiv a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}} > 0$ is a constant.

(b) From Hotelling's Lemma, $\frac{\partial \pi(p,w)}{\partial p} = y(p,w)$

$$\frac{\partial \pi(p,w)}{\partial w} = -x(p,w)$$

So $$p\frac{\partial \pi(p,w)}{\partial p} + w\frac{\partial \pi(p,w)}{\partial w} = py(p,w) - wx(p,w)$$

$$= \pi(p,w)$$

This just says that when you use the optimal input and output at the given prices (p,w), you get the maximum possible profit at (p,w).

Note: another way to think about this is that the profit function is linearly homogeneous. Any such function has the property that if you multiply each partial derivative by that argument of the function and add them all up, you get the original function.

(c) Again from Hotelling, $\frac{\partial \pi(p,w)}{\partial p} = y(p,w)$ so

$$\frac{\partial^2 \pi(p,w)}{\partial p^2} = \frac{\partial y(p,w)}{\partial p}$$

Doing the calculations, $y(p,w) = \left(\frac{1}{1-a}\right)p^{\frac{1}{1-a}-1} w^{-\frac{a}{1-a}} A$

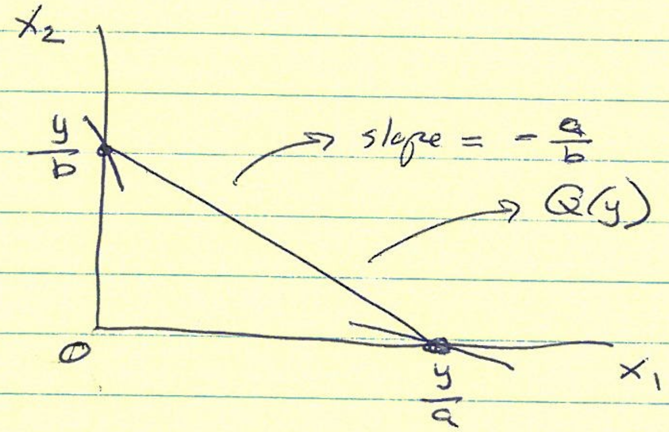
and $\frac{\partial y(p,w)}{\partial p} = \left(\frac{1}{1-a}\right)\left(\frac{a}{1-a}\right)p^{\frac{1}{1-a}-2} w^{-\frac{a}{1-a}} A > 0$

Therefore when $p \uparrow$, we have $y \uparrow$. From the production function, this implies $x \uparrow$.

Similarly, $\dfrac{\partial y(p,w)}{\partial w} = \left(\dfrac{1}{1-a}\right) p^{\frac{1}{1-a}-1} \left(\dfrac{-a}{1-a}\right) w^{\frac{-a}{1-a}-1} A < 0$

So when $w \uparrow$ we have $y \downarrow$. From the production function, this implies $x \downarrow$.

3 (a) It's easiest to think about this graphically, although you could also use Kuhn-Tucker methods.



If the slope of the isocost lines $-\frac{w_1}{w_2}$ is steeper than the isoquant, so ① $\frac{w_1}{w_2} > \frac{a}{b}$, the cost min solution is $x_1 = 0$, $x_2 = \frac{y}{b}$. ② If $\frac{w_1}{w_2} < \frac{a}{b}$, the solution is $x_1 = \frac{y}{a}$, $x_2 = 0$.

③ If $\frac{w_1}{w_2} = \frac{a}{b}$, any $(x_1, x_2) \geq 0$ on the isoquant is cost min.

In case ①, $c(w,y) = \dfrac{w_2 y}{b} = y \min\left\{\dfrac{w_1}{a}; \dfrac{w_2}{b}\right\}$ because $\dfrac{w_2}{b} < \dfrac{w_1}{a}$

in case ②, $c(w,y) = \dfrac{w_1 y}{a} = y \min\left\{\dfrac{w_1}{a}; \dfrac{w_2}{b}\right\}$ because $\dfrac{w_1}{a} < \dfrac{w_2}{b}$

In case ③, $c(w,y) = \dfrac{w_1 y}{a} = \dfrac{w_2 y}{b} = y \min\left\{\dfrac{w_1}{a}; \dfrac{w_2}{b}\right\}$

because <u>both</u> corners are optimal and $\dfrac{w_1}{a} = \dfrac{w_2}{b}$.

3(b) to min cost, set up the Lagrangean

$$L = w_1 x_1 + w_2 x_2 - \lambda \left[f(x) - y \right)$$

$$\Rightarrow \text{FOC:} \quad \left. \begin{aligned} w_1 - \lambda \frac{\partial f(x)}{\partial x_1} &= 0 \\ w_2 - \lambda \frac{\partial f(x)}{\partial x_2} &= 0 \end{aligned} \right\} \Rightarrow \frac{w_1}{w_2} = \frac{\frac{\partial f(x)}{\partial x_1}}{\frac{\partial f(x)}{\partial x_2}}$$

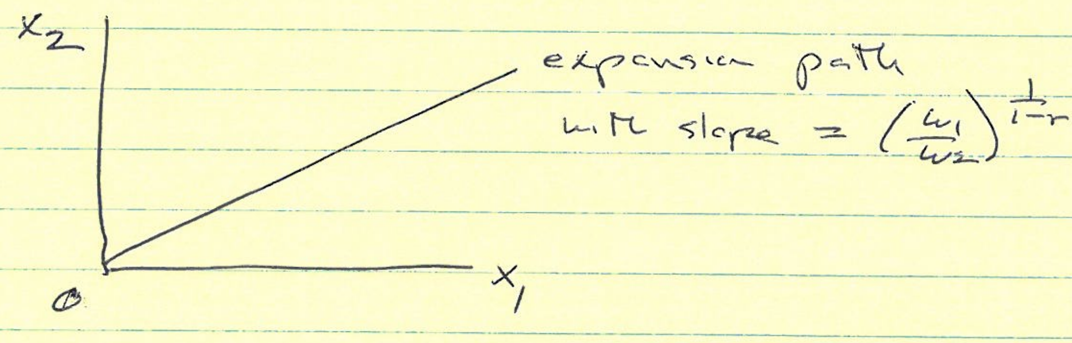
Compute the marginal products:

$$\frac{\partial f(x)}{\partial x_1} = \left(\frac{1}{r}\right) \left[x_1^r + x_2^r \right]^{\frac{1}{r}-1} (r) x_1^{r-1}$$

$$\frac{\partial f(x)}{\partial x_2} = \left(\frac{1}{r}\right) \left[x_1^r + x_2^r \right]^{\frac{1}{r}-1} (r) x_2^{r-1}$$

The ratio of the MPs is $\left(\frac{x_1}{x_2}\right)^{r-1}$. So along the EP we have

$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{r-1}$ or $\frac{x_2}{x_1} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{1-r}}$. This implies that the EP is a ray from the origin with a constant slope $\frac{x_2}{x_1}$ that does not depend on the output y:

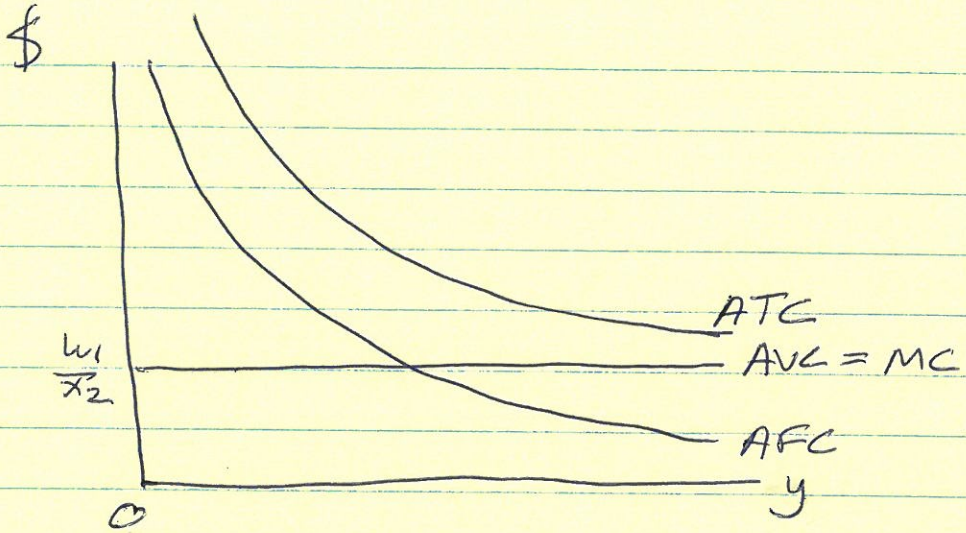


(c) In the short run $x_2 > 0$ is fixed, so to get output y the firm must use $x_1 = \frac{y}{x_2}$. Thus total cost is

$$C(w, y, x_2) = \underbrace{\frac{w_1 y}{x_2}}_{\text{variable}} + \underbrace{w_2 x_2}_{\text{fixed.}}$$

So $AFC = \frac{w_2 x_2}{y}$, $AVC = \frac{w_1}{x_2}$, $ATC = \frac{w_1}{x_2} + \frac{w_2 x_2}{y}$

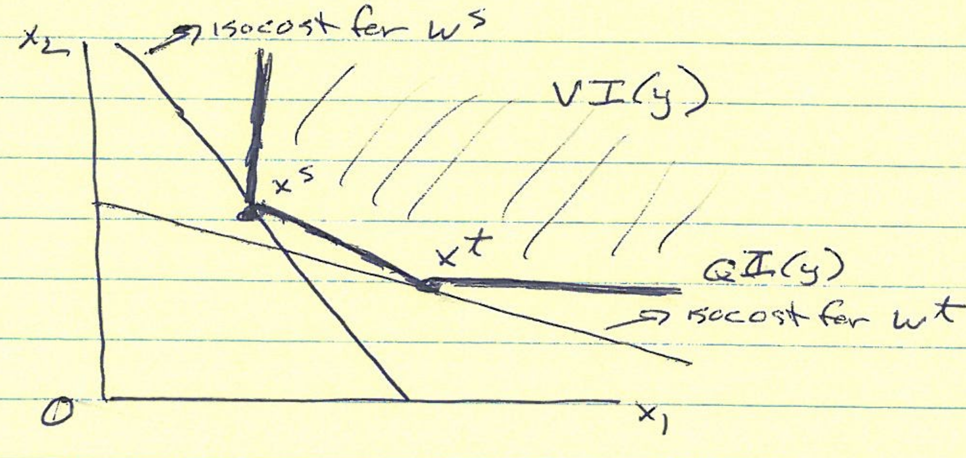
and $MC = \frac{w_1}{x_2}$



The firm's SR profit max problem can have a solution but only if $p \le \frac{w_1}{x_2} = MC$. If $p < \frac{w_1}{x_2}$, There is a unique solution with $y = 0$. If $p = \frac{w_1}{x_2}$ Then any $y \ge 0$ is a solution but The resulting profit is negative (due to fixed cost). If $p > \frac{w_1}{x_2}$ Then The firm can get arbitrarily large profit by choosing arbitrarily large y, so There is no solution.

The firm's LR profit max problem never has a solution because $y = x_1 x_2$ is a Cobb Douglas function where The exponents sum to 2, so it has increasing returns to scale.

4(a)

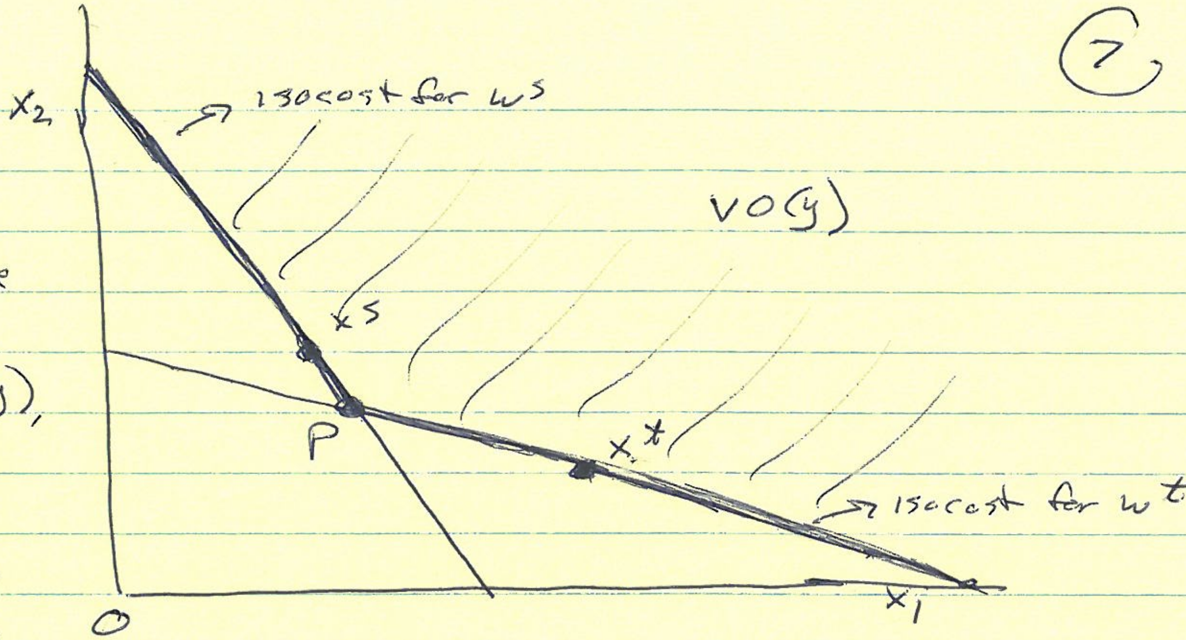


Convexity ⇒ line segment between x^s and x^t must be included.

Monotonicity ⇒ all points to The northeast of x^s, x^t, or The line segment between Them must be included.

Closed ⇒ The boundary $QI(y)$ must be included.

4(b)



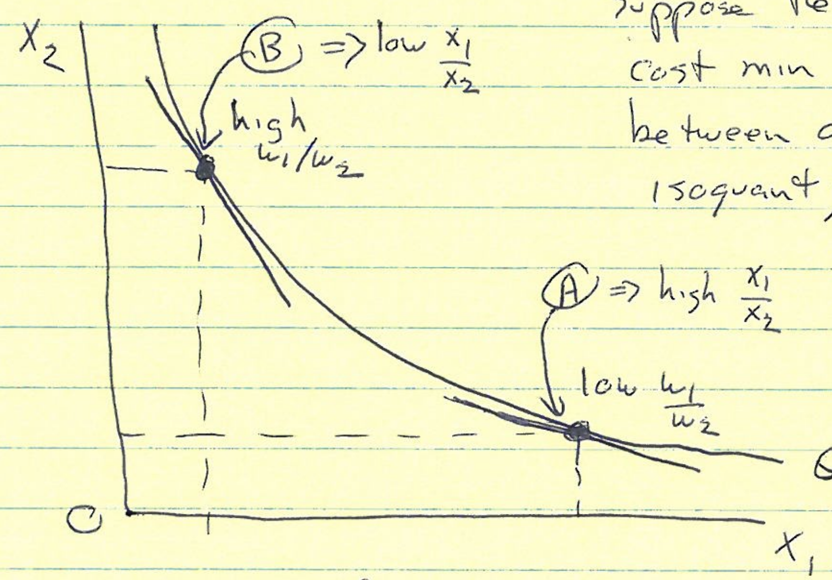
Heavy line segments give $QO(y)$, which is also the boundary of $VO(y)$.

No points below the isocost line through x^s can be included because any such point contradicts the fact that x^s is cost min at prices w^s.
Likewise no points below the isocost line through x^t can be included because any such point contradicts the fact that x^t is cost min at prices w^t.
~~The set of~~ All other points can be included in $VO(y)$. The resulting set is convex, monotone, and closed. In particular, it contains the boundary points in the isoquant set $QO(y)$.

(c) If the true set is $VI(y)$ then whenever the slope of the isocost line $-\frac{w_1}{w_2}$ is <u>steeper</u> than the line segment between x^s and x^t, the firm chooses x^s. If these slopes are <u>equal</u>, it is indifferent toward all points on this line segment.
If the isocost line is <u>flatter</u> than this line segment, the firm chooses x^t.

If the true set is $VO(y)$, whenever $-\frac{w_1}{w_2}$ gives an isocost line steeper than the one for w^s, the firm chooses a corner solution where the isocost line for w^s hits the vertical axis. If $-\frac{w_1}{w_2} = -\frac{w_1^s}{w_2^s}$ then the firm is indifferent toward all points between this corner solution and point P (see graph above). If $-\frac{w_1}{w_2}$ gives an isocost line flatter than the one for w^s but steeper than the one for w^t, the firm chooses point P. If $-\frac{w_1}{w_2} = -\frac{w_1^t}{w_2^t}$, then the firm is indifferent toward all points along the isocost line for w^t between P and the horizontal intercept. Finally, if $-\frac{w_1}{w_2}$ gives an isocost line flatter than the one for w^t, the firm chooses the corner solution where the isocost line for w^t hits the horizontal axis.

5(a) Fix some output $y > 0$ and consider the isoquant $Q(y)$. Assume for simplicity the input requirement set $V(y)$ is strictly convex, there are no boundary solutions, etc.



Suppose the student knows that cost min requires a tangency between an isocost line and the isoquant; she knows that the slope of the isocost line is $-\frac{w_1}{w_2}$; and she is willing to assume that TRS increases (so isoquant becomes steeper) as we move along the isoquant in the direction of higher $\frac{x_2}{x_1}$ (lower $\frac{x_1}{x_2}$).

Then as we move from low $\frac{w_1}{w_2}$ with a tangency point like A to higher $\frac{w_1}{w_2}$ with a tangency point like B, she should agree that the input quantity ratio $\frac{x_1}{x_2}$ falls.

This argument shows that for a fixed output level, and a cost-minimizing firm, there is an inverse relationship between $\frac{x_1}{x_2}$ and $\frac{w_1}{w_2}$, which we can write in the form

$$\frac{x_1}{x_2} = g\left(\frac{w_1}{w_2}\right)$$

Assuming differentiability, it is OK to take a derivative of this function when we define the elasticity of substitution.

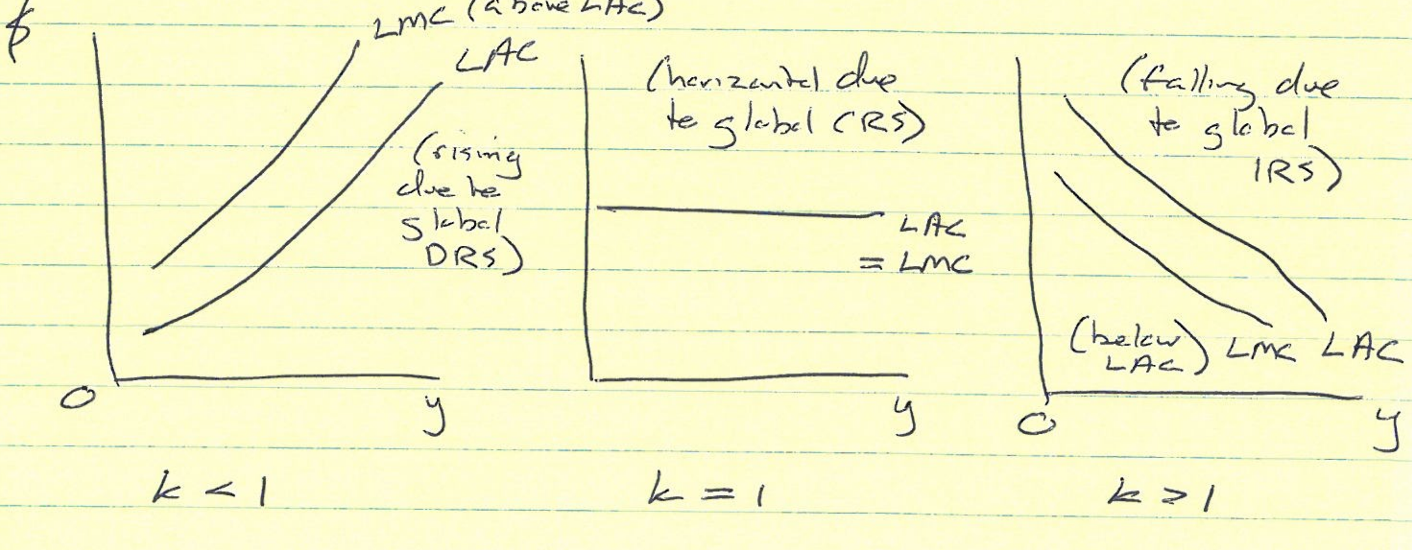
(b) $$e(x) = \frac{df(tx)}{dt} \cdot \frac{t}{f(tx)}\bigg|_{t=1}.$$

Suppose $f(x)$ is homogeneous of degree $k>0$. Then $f(tx) = t^k f(x)$. Substitute this into the definition of $e(x)$ so when we differentiate with respect to t, we get

$$\frac{df(tx)}{dt} = \frac{d}{dt}\left[t^k f(x)\right] = kt^{k-1}f(x).$$

Then evaluate everything at $t=1$ to get

$e(x) = k$. The resulting graphs are



5 (c) $c(w,y) = wx(w_i) - \lambda(w_i)\left[f(x(w_i)) - y\right]$

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w_i) + w\frac{\partial x}{\partial w_i} - \lambda\frac{\partial f}{\partial x}\frac{\partial x}{\partial w_i} - \frac{\partial \lambda}{\partial w_i}\left[f(x(w_i)) - y\right]$$

$$= x_i(w_i) + \underbrace{\left[w - \lambda\frac{\partial f}{\partial x}\right]}_{\substack{=0 \text{ by FOC} \\ \text{for cost min}}}\frac{\partial x}{\partial w_i} - \frac{\partial \lambda}{\partial w_i}\underbrace{\left[f(x(w_i)) - y\right]}_{\substack{=0 \text{ by FOC} \\ \text{for cost min}}}$$

So $\frac{\partial c(w,y)}{\partial w_i} = x_i(w_i)$

Interpretation. This is an alternative proof of Shepherd's Lemma. We have shown that the derivative of the cost function with respect to the price w_i is equal to the optimal quantity of input i.

Why does this work? The key idea is the envelope theorem. When we differentiate $c(w,y)$ with respect to the exogenous parameter w_i, we can ignore the indirect effects operating through $x(w_i)$ and $\lambda(w_i)$. The reason is that the firm has already optimized the levels of those variables, so the first-order effect on cost from the indirect effects is zero. Because these effects vanish, we are left only with the direct effect

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w)$$

where the rate at which cost goes up is equal to the quantity of the input the firm is currently using.